

# PROOF OF A CONJECTURE BY GAZEAU ET AL. USING THE GOULD HOPPER POLYNOMIALS

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**ABSTRACT.** We prove the “strong conjecture” expressed in [1] about the coefficients of the Taylor expansion of the exponential of a polynomial. This implies the “weak conjecture” as a special case. The proof relies mainly about properties of the Gould-Hopper polynomials.

## 1. INTRODUCTION

In [1], the authors state the following conjecture:

**Conjecture 1.** *If  $\{a_i, 2 \leq i \leq p\}$  are positive numbers, and with the notation  $x_n! = \prod_{k=1}^n x_k$ , then the numbers  $x_i$  such that*

$$\exp\left(t + \sum_{i=2}^p \frac{a_i}{i} t^i\right) = \sum_{n \geq 0} \frac{t^n}{x_n!}$$

*satisfy the recurrence relation*

$$x_n = \frac{n+1}{1 + \sum_{i=2}^p a_i \frac{x_n!}{x_{n-i+1}!}}.$$

In the following, we prove this conjecture using Gould-Hopper polynomials as defined in [2] and some integral representations of these polynomials as introduced in [3].

## 2. PRELIMINARY TOOLS

In [4], Nieto and Truax consider the operator

$$I_j = \exp\left[\left(c \frac{d}{dx}\right)^j\right]$$

where  $c$  is a constant and  $j$  an integer. They remark that, for any well-behaved function  $f$ ,  $I_1$  acts as the translation operator

$$I_1 f(x) = f(x+c),$$

which can also be viewed as the probabilistic expectation

$$I_1 f(x) = \mathbb{E}f(x + Z_1)$$

where  $Z_1$  is the deterministic variable equal to 1.

In the case  $j = 2$ , with  $Z_2$  denoting a Gaussian random variable with variance 2,  $I_2$  acts as the Gauss-Weierstrass transform

$$I_2 f(x) = \mathbb{E}f(x + cZ_2).$$

It was shown in [4] that this result can be extended to any integer value of  $j$  as follows:

**Proposition 2.** *For any integer  $j \geq 1$ , there exists a complex-valued random variable  $Z_j$  such that the following representation*

$$(2.1) \quad I_j = \mathbb{E}f(x + cZ_j)$$

*holds.*

The properties of the complex-valued random variable  $Z_j$  were studied further in [3]. The only important property we need to know here is that

$$\mathbb{E}Z_j^k = \begin{cases} 0 & \text{if } k \not\equiv 0 \pmod{j} \\ \frac{(pj)!}{p!} & \text{if } k = pj, p \in \mathbb{N} \end{cases}.$$

and that, as a consequence, its characteristic function

$$(2.2) \quad \mathbb{E} \exp(uZ_j) = \exp(u^j), \quad u \geq 0,$$

since a straightforward computation gives

$$(2.3) \quad \mathbb{E} \exp(uZ_j) = \sum_{k=0}^{+\infty} \frac{u^k}{k!} \mathbb{E}Z_j^k = \sum_{p=0}^{+\infty} \frac{u^{pj}}{pj!} \frac{pj!}{p!} = \exp(u^j).$$

**Definition 3.** The Gould-Hopper polynomials [2, p.58] are defined as

$$(2.4) \quad g_n^m(x, h) = \mathbb{E} \left( x + h^{\frac{1}{m}} Z_m \right)^n$$

and can be naturally generalized as

$$(2.5) \quad g_n(x, \mathbf{h}) = \mathbb{E} \left( x + \sum_{i=2}^p h_i^{\frac{1}{i}} Z_i \right)^n$$

for any vector  $\mathbf{h} = [h_2, \dots, h_p]$  such that  $\{h_i \geq 0, 2 \leq i \leq p\}$ .

**Lemma 4.** The Gould-Hopper polynomials satisfy the following identity

$$(2.6) \quad g_n(x, \mathbf{h}) = \exp \left( \sum_{i=2}^p h_i \frac{d^i}{dx^i} \right) x^n.$$

*Proof.* We have

$$\exp \left( \sum_{i=2}^p h_i \frac{d^i}{dx^i} \right) x^n = \prod_{i=2}^p \exp \left( h_i \frac{d^i}{dx^i} \right) x^n$$

and the result follows by applying successively (2.1), we deduce the result.  $\square$

**Lemma 5.** The generating function of the Gould-Hopper polynomials  $g_n(x, \mathbf{h})$  is

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!} g_n(x, \mathbf{h}) = \exp \left( xt + \sum_{i=2}^p h_i t^i \right).$$

*Proof.* From the definition (2.4), we deduce, with  $Z_2, \dots, Z_p$  as in Proposition 2,

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{t^n}{n!} g_n(x, \mathbf{h}) &= \mathbb{E} \exp \left( t \left( x + \sum_{i=2}^p h_i^{\frac{1}{i}} Z_i \right) \right) \\ &= \exp(xt) \prod_{i=2}^p \mathbb{E} \exp \left( t h_i^{\frac{1}{i}} Z_i \right) \end{aligned}$$

and the result follows from (2.3).  $\square$

From this lemma, we deduce that the factorial coefficients  $x_n!$  satisfy

$$(x_n!)^{-1} = \frac{g_n(x, \mathbf{h})}{n!}.$$

In order to obtain a recurrence formula for the numbers  $x_n$ , we need the following recurrence relation on the Gould-Hopper polynomials.

**Lemma 6.** *The Gould-Hopper polynomials (2.6) satisfy the difference equation*

$$g_{n+1}(x, \mathbf{h}) = xg_n(x, \mathbf{h}) + \sum_{k=2}^p kh_k \frac{n!}{(n-k+1)!} g_{n+1-k}(x, \mathbf{h}).$$

*Proof.* The moment representation (2.5) yields

$$\begin{aligned} g_{n+1}(x, \mathbf{h}) &= \mathbb{E} \left( x + \sum_{i=2}^p h_i^{\frac{1}{i}} Z_i \right)^{n+1} = \mathbb{E} \left( x + \sum_{i=2}^p h_i^{\frac{1}{i}} Z_i \right) \left( x + \sum_{i=2}^p h_i^{\frac{1}{i}} Z_i \right)^n \\ &= x \mathbb{E} \left( x + \sum_{i=2}^p h_i^{\frac{1}{i}} Z_i \right)^n + \sum_{k=2}^p h_k^{\frac{1}{k}} \mathbb{E} Z_k \left( x + \sum_{i=2}^p h_i^{\frac{1}{i}} Z_i \right)^n. \end{aligned}$$

The first term is identified as  $xg_n(x, \mathbf{h})$  and the second term is computed using the following lemma.  $\square$

**Lemma 7.** *The random variables  $Z_j$  as defined in Proposition 2 satisfy the following Stein identity*

$$\mathbb{E} \left( Z_k f \left( x + \sum_{i=2}^p h_i^{\frac{1}{i}} Z_i \right) \right) = kh_k^{1-\frac{1}{k}} \mathbb{E} f^{(k-1)} \left( x + \sum_{i=2}^p h_i^{\frac{1}{i}} Z_i \right)$$

for any smooth function  $f$ .

*Proof.* The partial derivative

$$\frac{\partial}{\partial h_k} \mathbb{E} \left( f \left( x + \sum_{i=2}^p h_i^{\frac{1}{i}} Z_i \right) \right) = \mathbb{E} \left( Z_k f' \left( x + \sum_{i=2}^p h_i^{\frac{1}{i}} Z_i \right) \right) \frac{1}{k} h_k^{\frac{1}{k}-1}$$

can also be computed from (2.6) as

$$\begin{aligned} \frac{\partial}{\partial h_k} \exp \left( \sum_{i=2}^p h_i \frac{d^i}{dx^i} \right) f(x) &= \exp \left( \sum_{i=2}^p h_i \frac{d^i}{dx^i} \right) \frac{d^k}{dx^k} f(x) \\ &= \exp \left( \sum_{i=2}^p h_i \frac{d^i}{dx^i} \right) f^{(k)}(x) = \mathbb{E} f^{(k)} \left( x + \sum_{i=2}^p h_i^{\frac{1}{i}} Z_i \right) \end{aligned}$$

so that

$$\mathbb{E} \left( Z_k f' \left( x + \sum_{i=2}^p h_i^{\frac{1}{i}} Z_i \right) \right) \frac{1}{k} h_k^{\frac{1}{k}-1} = \mathbb{E} f^{(k)} \left( x + \sum_{i=2}^p h_i^{\frac{1}{i}} Z_i \right)$$

which is the result after replacing  $f'$  by  $f$  in both sides. Using this result yields with  $f(x) = x^n$  yields the proof of Lemma 6.  $\square$

We can now prove the Conjecture 1 as follows: by Lemma 6, the quantities

$$(x_n!)^{-1} = \frac{1}{n!} g_n(x, \mathbf{h})$$

satisfy the recurrence

$$(n+1)(x_{n+1}!)^{-1} = x(x_n!)^{-1} + \sum_{k=2}^p kh_k (x_{n+1-k}!)^{-1}.$$

Dividing both sides by  $(x_n!)^{-1}$ , and remarking that

$$x_{n+1}^{-1} = \frac{(x_{n+1}!)^{-1}}{(x_n!)^{-1}},$$

we deduce

$$(n+1)x_{n+1}^{-1} = x + \sum_{k=2}^p kh_k \frac{x_n!}{x_{n+1-k}!}.$$

Choosing  $h_k = \frac{ak}{k}$  and  $x = 1$ , we deduce the result.

We note that we have a proved slightly more general result than Conjecture 1, namely the fact that the coefficients  $x_n$  in the expression

$$\exp\left(xt + \sum_{i=2}^p \frac{a_i}{i} t^i\right) = \sum_{n \geq 0} \frac{t^n}{x_n!}$$

satisfy the recurrence

$$x_{n+1} = \frac{n+1}{x + \sum_{k=2}^p a_k \frac{x_n!}{x_{n+1-k}!}}.$$

#### REFERENCES

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